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Source: *Isis*, Vol. 69, No. 4 (Dec., 1978), pp. 489-513
Published by: The University of Chicago Press on behalf of The History of Science Society
Stable URL: http://www.jstor.org/stable/231089

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*By M. F. Burnyeat**

INTRODUCTION

THERE IS A WELL-KNOWN PASSAGE in Plato's *Theaetetus* (147d–148b) where the young Theaetetus recounts a story which has figured in histories of mathematics ever since. The story concerns a geometry lesson in which Theodorus of Cyrene gave separate case-by-case proofs that the side of a square with area 3 square feet, 5 square feet, and so on up to 17 square feet, is incommensurable with the side of a 1-foot (unit) square; whereupon Theodorus' pupils—Theaetetus and a companion of his called Socrates the Younger—formulated a general definition of the important mathematical notion of linear incommensurability. The question is, what kind of evidence, if any, does the story provide for actual historical developments in Greek mathematics?

It has been traditional among historians of mathematics to suppose that Plato's scene celebrates Theaetetus' part in a historical reality, a decisive advance in the theory of irrationals made, no doubt, in Theaetetus' adulthood but projected back into his student days in order to fit the dramatic circumstances of the dialogue. Dramatically, the dialogue is set in 399 B.C.: Socrates is in the last year of his life, with the prospect of his trial and condemnation already looming (142c, 210d); Theaetetus is depicted as a mere youth of sixteen or even less (142c, 143e, 168d, *et al.*), unbearded (168e), and with some growing still to do (155b), while Theodorus is a distinguished old man (143de, 146b) of around sixty or seventy years.[1] If, then, as historians suppose, Theaetetus took the theory of irrationals forward from the stage to which Theodorus had brought it, on grounds of their respective ages neither Theodorus' contribution nor Theaetetus' is likely to have been made very near the time of the lesson. The story is a fiction, devised by Plato for his own purposes in the dialogue.

Received September 1976; revised/accepted November 1976.
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I was enabled to undertake the research for this paper by my tenure of a Radcliffe Fellowship. I am grateful to the Radcliffe Trust for the gift of the fellowship and to University College London for allowing me to take it up. I should like also to acknowledge the benefit of criticisms and suggestions from James Dybikowski, Ian Mueller, Malcolm Schofield, and Richard Sorabji.

[1]For estimates of the life times of Theaetetus and Theodorus, based on the *Theaetetus* and such other evidence as there is, see Eva Sachs, *De Theaeteto Atheniensi mathematico* (Dissertation, Berlin, 1914), Ch. 1; Kurt von Fritz in Pauly-Wissowa, *Realencyclopädie der classischen Altertumswissenschaft,* Vol. V.A.2. (Stuttgart: J. B. Metzler, 1934), *s.v.* "Theaitetos" (art. 2), pp. 1351–1352, *s.v.* "Theodoros" (art. 31), pp. 1811–1812.

The question now becomes the following: is it reasonable to think that Plato's purpose in composing the story includes the celebration of an important development in the progress of mathematics? Historians have not hesitated to affirm that it is. Two kinds of considerations support their judgment. The first is evidence independent of Plato that Theaetetus did contribute decisively to the theory of irrationals (see below); the second is evidence that Plato did intend the *Theaetetus* to mark Theaetetus' mathematical achievements.

The main body of the dialogue is prefaced by a miniature conversation between two Megarian philosophers, imagined as taking place just after a battle near Corinth in 369 B.C.[2] Theaetetus, who took part in the fighting, is now dying from wounds and dysentery. His conduct on the battlefield is commended: it was quite as would be expected from a man of his virtues. Then it is recalled that Socrates had been greatly impressed when he met him in 399 (a reference to the discussion which constitutes the dialogue proper) and predicted for him a distinguished career (142ad). It is natural to understand Socrates' prediction as Plato's testimony to Theaetetus' actual intellectual achievements. The prefatory encomium on the dying mathematician is Plato dedicating the dialogue to the memory of a friend and colleague in the Academy.

That being so, it seems equally natural to find something of Theaetetus' mathematical accomplishments prefigured in the dialogue itself, in the story we began with. The geometry lesson is fictitious, but if we have read the signs of Plato's purpose correctly, it may still encapsulate real contributions by master and pupil. In the words of van der Waerden, "It can not have been Plato's intention to give credit to Theodorus for what is due to Theaetetus, nor vice versa."[3]

So stands the traditional interpretation. To mention but a few of the distinguished names who have adhered to its reading of Plato's purpose: Heinrich Vogt, Eva Sachs, Sir Thomas Heath, Kurt von Fritz, B. L. van der Waerden, and Siegfried Heller, although differing in important details of their reconstruction, all agree that we are dealing with an original contribution to science by Theaetetus, building on results previously attained by Theodorus.[4] Ancient scholarship took the same view, as we shall see. Recently, however, a dissentient voice has been heard. In a series of writings Árpád Szabó has maintained that in the scene before us Theaetetus accomplishes nothing of worth or even interest. If, at first reading, the passage suggests otherwise, that is because Theaetetus himself, who tells the story, naïvely thinks he has made a discovery, when really he has been deliberately led by Theodorus to work out in the

[2]For the identification and dating of the battle, see Sachs, *De Theaeteto,* pp. 22–40; Auguste Diès in the Budé edition of the *Theaetetus,* Platon, *Oeuvres complètes,* Vol. VIII.2 (Paris: Association Guillaume Budé, 1924), pp. 120–123.

[3]B. L. van der Waerden, *Science Awakening,* trans. Arnold Dresden (Groningen: Noordhoff, 1954), p. 142; cf. the similar sentiment in Kurt von Fritz, "The Discovery of Incommensurability by Hippasus of Metapontum," *Annals of Mathematics,* 1945, *46*:242–264, here cited from David J. Furley and R. E. Allen, eds., *Studies in Presocratic Philosophy,* Vol. I (London: Routledge & Kegan Paul; New York: Humanities Press, 1969), p. 384.

[4]Heinrich Vogt, "Die Entdeckungsgeschichte des Irrationalen nach Plato und anderen Quellen des 4. Jahrhunderts," *Bibliotheca Mathematica,* 1909–1910, 3 F., *10*:97–155; Sachs, *De Theaeteto;* Thomas L. Heath, *A History of Greek Mathematics* (Oxford: Clarendon Press, 1921), Vol. I, pp. 202 ff.; von Fritz, "Theaitetos" and "Theodoros"; B. L. van der Waerden, "Die Arithmetik der Pythagoreer," *Mathematische Annalen,* 1947–1949, *120*:127–153, 676–700, here cited from Oskar Becker, ed., *Zur Geschichte der griechischen Mathematik* (Darmstadt: Wissenschaftliche Buchgesellschaft, 1965); van der Waerden, *Science Awakening;* Siegfried Heller, "Ein Beitrag zur Deutung der Theodoros-Stelle in Platons Dialog 'Theaetet,'" *Centaurus,* 1956–1958, *5*:1–58.

lesson a point or two long familiar to mathematicians and already implicit in his teacher's instruction.[5]

Szabó's interpretation is elaborately argued, but in advance of any confrontation with his arguments it can be seen to jar with Plato's presentation of Theaetetus' definition of incommensurability. In no other dialogue does Socrates' interlocutor come forward with a contribution of his own to compare with the one displayed here, and it is greeted by Socrates with praise on two significant counts: as an exemplary answer amply justifying the portrait Theodorus had given earlier of his pupil's outstanding talent (148b, referring to 144ab),[6] and as a model definition which Theaetetus would do well to follow in pursuing the task Socrates has set him of explaining what knowledge is (148d). Evidently, the definition is a notable achievement. But at what level? Is it a fundamental contribution to science or just a token of student prowess—or something in between? The traditional interpretation favors the first, Szabó the second; but perhaps these are not the exhaustive alternatives they are usually assumed to be. That is the possibility I propose to explore here.

I take as my starting assumption that, as already indicated, the geometry lesson is not a detachable historical report but a fiction devised by Plato for his own purposes in the dialogue. Hence the historical import of the lesson must be evaluated by a reading which takes account of its context in the dialogue and asks what philosophical point Plato designed it to make. Szabó's dismissive estimate is not to be believed, for the reason already given and others to be added, but his interpretation is a challenge to go over the scene once more with a view to making better philosophical sense of Theaetetus' mathematics. Placed as it is before the dialogue's inquiry into the nature of knowledge and after the generous praise of Theaetetus' talent and promise (in the preface at 142bd and in Theodorus' portrait of him at 143e–144b), the passage needs to be read with a sense of what the historical Theaetetus and his mathematical achievements might have meant to Plato and his contemporaries. That is how ancient scholars tried to read it, and I shall have regard for what can be learned from their discussions, which were more extensive than has usually been appreciated.

THE TEXT

As a basis for discussion I reproduce the text of 147c 7–148d 7 followed by the admirable English of John McDowell.[7] Refinements on points of translation can be postponed until they are needed.

[5] Árpád Szabó, "Der mathematische Begriff δύναμις und das sog. 'geometrische Mittel,'" *Maia,* 1963, *15*:219–256; "Theaitetos und das Problem der Irrationalität in der griechischen Mathematikgeschichte," *Acta Antiqua Academiae Scientiarum Hungaricae,* 1966, *14*:303–358; *Anfänge der griechischen Mathematik* (Munich/Vienna: Oldenbourg, 1969). Since there is much overlapping and repetition among these, I shall cite *Anfänge* except in cases of some positive difference. Szabó's interpretation is accepted as "on the whole, sound" by Jaap Mansfeld, "Notes on Some Passages in Plato's Theaetetus and in the 'Anonymous Commentary,'" in *Zetesis: Festschrift E. de Strycker* (Antwerp/Utrecht: De Nederlandsche Boekhandel, 1973), p. 112. For a less favorable scholarly judgment, see Walter Burkert's review of Szabó, *Anfänge,* in *Erasmus,* 1971, *23*:102–105. But I have not found in the literature any large-scale critical examination of Szabó's views.

[6] Of the traits listed by Theodorus the relevant one here is no doubt Theaetetus' quickness of mind (144a 6–7: οἵ τε ὀξεῖς ὥσπερ οὗτος καὶ ἀγχίνοι). Aristotle, *Posterior Analytics* I 34, explains ἀγχίνοια as quickness at hitting upon the explanation of something. In Plato's *Laws* (747b) it is claimed that quickness of mind, facility at learning, and good memory, which are the three intellectual qualities mentioned in the portrait of Theaetetus, are especially promoted by a mathematical education.

[7] *Platonis opera,* ed. J. Burnet (2nd ed., Oxford: Clarendon Press, 1905), Vol. I. John McDowell, *Plato Theaetetus* (Oxford: Clarendon Press, 1973).

ΘΕΑΙ. Ῥᾴδιον, ὦ Σώκρατες, νῦν γε οὕτω φαίνεται· ἀτὰρ
κινδυνεύεις ἐρωτᾶν οἷον καὶ αὐτοῖς ἡμῖν ἔναγχος εἰσῆλθε
διαλεγομένοις, ἐμοί τε καὶ τῷ σῷ ὁμωνύμῳ τούτῳ Σωκράτει. d
ΣΩ. Τὸ ποῖον δή, ὦ Θεαίτητε;
ΘΕΑΙ. Περὶ δυνάμεών τι ἡμῖν Θεόδωρος ὅδε ἔγραφε, τῆς
τε τρίποδος πέρι καὶ πεντέποδος ἀποφαίνων ὅτι μήκει οὐ
σύμμετροι τῇ ποδιαίᾳ, καὶ οὕτω κατὰ μίαν ἑκάστην προαιρού-
μενος μέχρι τῆς ἑπτακαιδεκάποδος· ἐν δὲ ταύτῃ πως ἐνέσχετο.
ἡμῖν οὖν εἰσῆλθέ τι τοιοῦτον, ἐπειδὴ ἄπειροι τὸ πλῆθος αἱ
δυνάμεις ἐφαίνοντο, πειραθῆναι συλλαβεῖν εἰς ἕν, ὅτῳ πάσας
ταύτας προσαγορεύσομεν τὰς δυνάμεις. e
ΣΩ. Ἦ καὶ ηὕρετέ τι τοιοῦτον;
ΘΕΑΙ. Ἔμοιγε δοκοῦμεν· σκόπει δὲ καὶ σύ.
ΣΩ. Λέγε.
ΘΕΑΙ. Τὸν ἀριθμὸν πάντα δίχα διελάβομεν· τὸν μὲν
δυνάμενον ἴσον ἰσάκις γίγνεσθαι τῷ τετραγώνῳ τὸ σχῆμα
ἀπεικάσαντες τετράγωνόν τε καὶ ἰσόπλευρον προσείπομεν.
ΣΩ. Καὶ εὖ γε.
ΘΕΑΙ. Τὸν τοίνυν μεταξὺ τούτου, ὧν καὶ τὰ τρία καὶ
τὰ πέντε καὶ πᾶς ὃς ἀδύνατος ἴσος ἰσάκις γενέσθαι, ἀλλ' ἢ 148
πλείων ἐλαττονάκις ἢ ἐλάττων πλεονάκις γίγνεται, μείζων
δὲ καὶ ἐλάττων ἀεὶ πλευρὰ αὐτὸν περιλαμβάνει, τῷ προμήκει
αὖ σχήματι ἀπεικάσαντες προμήκη ἀριθμὸν ἐκαλέσαμεν.
ΣΩ. Κάλλιστα. ἀλλὰ τί τὸ μετὰ τοῦτο;
ΘΕΑΙ. Ὅσαι μὲν γραμμαὶ τὸν ἰσόπλευρον καὶ ἐπίπεδον
ἀριθμὸν τετραγωνίζουσι, μῆκος ὡρισάμεθα, ὅσαι δὲ τὸν ἑτερο-
μήκη, δυνάμεις, ὡς μήκει μὲν οὐ συμμέτρους ἐκείναις, τοῖς δ' b
ἐπιπέδοις ἃ δύνανται. καὶ περὶ τὰ στερεὰ ἄλλο τοιοῦτον.
ΣΩ. Ἄριστά γ' ἀνθρώπων, ὦ παῖδες· ὥστε μοι δοκεῖ ὁ
Θεόδωρος οὐκ ἔνοχος τοῖς ψευδομαρτυρίοις ἔσεσθαι.
ΘΕΑΙ. Καὶ μήν, ὦ Σώκρατες, ὅ γε ἐρωτᾷς περὶ ἐπι-
στήμης οὐκ ἂν δυναίμην ἀποκρίνασθαι ὥσπερ περὶ τοῦ
μήκους καὶ τῆς δυνάμεως. καίτοι σύ γέ μοι δοκεῖς τοιοῦτόν
τι ζητεῖν· ὥστε πάλιν αὖ φαίνεται ψευδὴς ὁ Θεόδωρος.
ΣΩ. Τί δέ; εἴ σε πρὸς δρόμον ἐπαινῶν μηδενὶ οὕτω c
δρομικῷ ἔφη τῶν νέων ἐντετυχηκέναι, εἶτα διαθέων τοῦ
ἀκμάζοντος καὶ ταχίστου ἡττήθης, ἧττόν τι ἂν οἴει ἀληθῆ
τόνδ' ἐπαινέσαι;
ΘΕΑΙ. Οὐκ ἔγωγε.
ΣΩ. Ἀλλὰ τὴν ἐπιστήμην, ὥσπερ νυνδὴ ἐγὼ ἔλεγον,
σμικρόν τι οἴει εἶναι ἐξευρεῖν καὶ οὐ τῶν πάντῃ ἄκρων;
ΘΕΑΙ. Νὴ τὸν Δί' ἔγωγε καὶ μάλα γε τῶν ἀκροτάτων.
ΣΩ. Θάρρει τοίνυν περὶ σαυτῷ καὶ τὶ οἴου Θεόδωρον
λέγειν, προθυμήθητι δὲ παντὶ τρόπῳ τῶν τε ἄλλων πέρι καὶ d
ἐπιστήμης λαβεῖν λόγον τί ποτε τυγχάνει ὄν.
ΘΕΑΙ. Προθυμίας μὲν ἕνεκα, ὦ Σώκρατες, φανεῖται.
ΣΩ. Ἴθι δή—καλῶς γὰρ ἄρτι ὑφηγήσω—πειρῶ μιμού-
μενος τὴν περὶ τῶν δυνάμεων ἀπόκρισιν, ὥσπερ ταύτας
πολλὰς οὔσας ἑνὶ εἴδει περιέλαβες, οὕτω καὶ τὰς πολλὰς
ἐπιστήμας ἑνὶ λόγῳ προσειπεῖν.

THEAETETUS. It [*sc.* to say what knowledge is] looks easy now, Socrates, when you put it like that. There's a point that came up in a discussion I was having recently with your
d namesake, Socrates here; it rather seems that what you're asking for is something of the same sort.

SOCRATES. What sort of point was it, Theaetetus?

THEAETETUS. Theodorus here was drawing diagrams to show us something about powers—namely that a square of three square feet and one of five square feet aren't commensurable, in respect of length of side, with a square of one square foot; and so on, selecting each case individually, up to seventeen square feet. At that point he somehow got tied up. Well, since the powers seemed to be unlimited in number, it occurred to us to do
e something on these lines: to try to collect the powers under one term by which we could refer to them all.

SOCRATES. And did you find something like that?

THEAETETUS. I think so; but you must look into it too.

SOCRATES. Tell me about it.

THEAETETUS. We divided all the numbers into two sorts. If a number can be obtained by multiplying some number by itself, we compared it to what's square in shape, and called it square and equal-sided.

SOCRATES. Good.

THEAETETUS. But if a number comes in between—these include three and five, and in
148 fact any number which can't be obtained by multiplying a number by itself, but is obtained by multiplying a larger number by a smaller or a smaller by a larger, so that the sides containing it are always longer and shorter—we compared it to an oblong shape, and called it an oblong number.

SOCRATES. Splendid. But what next?

THEAETETUS. We defined all the lines that square off equal-sided numbers on plane surfaces as lengths, and all the lines that square off oblong numbers as powers, since they
b aren't commensurable with the first sort in length, but only in respect of the plane figures which they have the power to form. And there's another point like this one in the case of solids.

SOCRATES. That's absolutely excellent, boys. I don't think Theodorus is going to be up on a charge of perjury.

THEAETETUS. Still, Socrates, I wouldn't be able to answer your question about knowledge in the way we managed with lengths and powers. But it seems to me to be something of that sort that you're looking for. So Theodorus does, after all, turn out to have said something false.

c SOCRATES. But look here, suppose he'd praised you for running, and said he'd never come across a young man who was so good at it; and then you'd run a race and been beaten by the fastest starter, a man in his prime. Do you think his praise would have been any less true?

THEAETETUS. No.

SOCRATES. And what about knowledge? Do you think it's a small matter to seek it out, as I was saying just now—not one of those tasks which are arduous in every way?

THEAETETUS. Good heavens, no: I think it's really one of the most arduous of tasks.

SOCRATES. Well then, don't lose heart about yourself, and accept that there was
d something in what Theodorus said. Always do your best in every way; and as for knowledge, do your best to get hold of an account of what, exactly, it really is.

THEAETETUS. If doing my best can make it happen, Socrates, it will come clear.

SOCRATES. Come on, then—because you've just sketched out the way beautifully—try to imitate your answer about the powers. Just as you collected them, many as they are, in one class, try, in the same way, to find one account by which to speak of the many kinds of knowledge.

INTERPRETATIVE PARAPHRASE

The bare facts of the case, as Theaetetus tells them, are as follows. He and a companion, the younger Socrates,[8] were sparked off by a series of incommensurability proofs undertaken by Theodorus. Theodorus showed them that, given squares of area 1 square foot and 3 square feet respectively, the side of the latter is incommensurable (has no common measure) with the side of the former, that is, with the 1-foot (unit) length. We are not told how he proved this result, only that he did the same for a square 5 feet in area and proceeded case by case as far as the square of 17 feet. In effect, since the sides of these squares are $\sqrt{3}$ feet, $\sqrt{5}$ feet, . . . $\sqrt{17}$ feet, he proved the irrationality of the square roots of each of the integers between 3 and 17 (with the exception, naturally, of 4, 9, and 16, which have integral square roots), but that is not how he expressed it. Greek mathematics recognized no numbers but the natural numbers (positive integers) from 1, or often only 2, onward and treated of irrational quantities as geometrical entities, in this instance, lines identified by the areas of the squares that can be constructed on them. Hence the exclusively geometrical form taken by Theodorus' lesson.

Now Theaetetus and the younger Socrates had the idea to attempt a general characterization of these magnitudes. They did it with the help of a division of numbers (meaning, as explained, the positive integers) into two classes: *square numbers* are those numbers which can be resolved into equal factors (as, e.g., 4 is 2 × 2), being so called by way of comparison with equal-sided figures of that shape; *oblong numbers* are all the rest, so called because they can only be resolved into unequal factors. The boys could then define *length* (the word is used here as a label for commensurable lengths) = side of (a square with area given by) a square number; and *power* (the term used for incommensurable lengths) = side of (a square with area given by) an oblong or nonsquare number. This last, the object of the exercise, amounts to saying that for any positive integer n, $\sqrt{n}$ is irrational if and only if there is no positive integer m such that $n = m \times m$; although, once again, that is not how a Greek thought of it. Finally, Theaetetus explains that they called the incommensurable lines "powers"—in effect, "square lines" as opposed to "length lines" (see below)—because they are incommensurable with the "lengths" in length but commensurable with them in respect of the squares they have the power to form; and he indicates that analogous distinctions can be defined for solids, to deal with (what we call) rational and irrational cube roots.

The remainder of the passage quoted (148bd) is a tailpiece needing no special elucidation but containing several important clues for evaluating Theaetetus' story. With that in mind and the full text before us, we can make a start on problems of interpretation.

FIRST PROBLEM: THE EXTENT OF THEAETETUS' ORIGINALITY

It is an obvious point that 2 should be the first oblong number. If Theaetetus mentions 3 as his first example (147e 9), that is presumably because it was with the 3-

[8]This Socrates, since he is referred to as present (147d 1), must be one of the group of friends with whom Theaetetus entered at the very beginning of the dialogue (144c; cf. 168d). In the *Sophist* (216a, 218b) he returns with Theodorus and Theaetetus for the sequel conversation which continues into the *Politicus*, where he takes over from Theaetetus as respondent (*Politicus* 257c–258a). Little is known about him. A reference in Aristotle, *Metaphysics* Z 11, on which cf. E. Kapp, "Sokrates der Jüngere," *Philologus*, 1924, *79*:228–233, implies mathematico-philosophical interests typical of the Academy; that he was in the Academy, and had political interests as well, is confirmed by the way he is spoken of by the author of the eleventh of the Platonic *Epistles* (358d).

foot square that Theodorus began his demonstrations (147d 4). (Why Theodorus should have done so is a question we will come to in due course.) But what of the square-oblong division itself? Szabó alleges—and the accusation is designed to shake our confidence in the whole story of the boys' discovery—that Theaetetus claims the division of numbers as his own and his friend's when it is nothing of the sort, but common property of Pythagorean origin.[9] Now what is and what is not to be credited to Pythagorean efforts in the field of mathematics is a difficult and highly disputed matter,[10] but there is no need to press the point. The fact is, it is by no means clear that under Greek conventions in these matters, which were less punctilious than ours, Theaetetus would be understood to claim ownership of the division or even of the terminology applied with it. No one familiar with the character of the Platonic Socrates supposes he is setting himself up as an original mathematician when he says at *Euthyphro* 12d that he would distinguish even numbers as those which are not scalene but isosceles. This must be an allusion to some method—obscure to us, though not, apparently, to Euthyphro—of representing numbers by triangles, but Socrates makes no reference to the mathematicians who devised it; he simply states that that is how, if he were asked to distinguish even numbers from odd, he would do it.

So too with the boys' division of numbers into square and oblong. Theaetetus' report (147e 5), "We divided all the numbers into two sorts," is plain narrative, staking no claim for originality. Socrates commends the procedure (147e 8, 148a 5), but his "What next?" at 148a 5 shows he sees it as subsidiary to the definition of commensurable and incommensurable still to come. The case of that definition is presented in a markedly different light: here is something the boys themselves thought of seeking, and found (147d 7, e 2–3), and it is this alone that is subsequently held up as their model answer (148b,d). Further, it is specifically this finding that confirms to Socrates the truth of Theodorus' testimony about Theaetetus (148b).[11]

It may seem that at 148c Socrates tempers his high praise of the definition by conceding that Theaetetus might not lead the field against the very toughest adult competition, insisting that he has, nevertheless, done brilliantly for a youngster. This indication of some junior status for the definition is certainly to be reckoned with (see below), but it does not swing the balance all the way to the mere schoolboy exercise of Szabó's interpretation. In context, the remark is Socrates' reassuring response to a doubt Theaetetus has expressed (148b) that he could deal as competently with the question "What is knowledge?" The intention is to encourage him to tackle this problem—a supremely difficult task (148c 7)—not to diminish in any way his earlier accomplishment with the mathematical one. That remains, if the story is read carefully, a finding, a discovery, meriting Socrates' unequivocal commendation.

SECOND PROBLEM: THEAETETUS' TERMINOLOGY

I turn now to a vexed issue of terminology. The word δύναμις ("power") occurs in what appears to be two distinct uses in Theaetetus' story. At the beginning of the passage (147de) it specifies what Theodorus' demonstrations were about; at the end

[9]Szabó, *Anfänge,* pp. 87–88, 106, following A. Frajese, "Perchè Teodoro di Cirene tralasciò la radice di due?," *Peridico di Mathematiche,* 1966, S.4, *44*:422.

[10]For a cautionary assessment, see Walter Burkert, *Lore and Science in Ancient Pythagoreanism,* trans. Edwin L. Minar, Jr. (Cambridge, Mass.: Harvard University Press, 1972), Ch. 6.

[11]This last point is well noted by Vogt, "Entdeckungsgeschichte," p. 113, contrasting it with the division of numbers, which he ascribes to the Pythagoreans. Szabó, *Anfänge,* p. 88, quotes Vogt on the latter, not the former.

(148b) it is defined to denominate incommensurable lines in opposition to the term μῆκος ("length"), which is restricted to commensurable lines. The defined use of δύναμις is a special one, for with a single possible exception (see below) there is no trace elsewhere of δύναμις or μῆκος being used in accordance with the boys' definition. By contrast, the initial occurrences of δύναμις must be supposed to carry a standard meaning which a reader could pick up without special guidance, since none is given. There are thus two problems: what does δύναμις signify at the start of the story, and how does this presumably standard use of the term relate to the defined use at the end? On his answer to these questions Szabó rests the main weight of his case. I shall argue that even if he is largely right about δύναμις, that is not good reason to accept his dismissive estimate of the episode before us.

I said that with one possible exception there is no trace elsewhere of the terms δύναμις or μῆκος being used in accordance with the boys' definition. Euclid employs only the complex descriptions "commensurable in length" (μήκει σύμμετρος—as here 147d 4–5, 148b 1) and "commensurable in square/power" (δυνάμει σύμμετρος; see e.g. *Elements* X defs. 2 and 3). In particular, we shall see that δύναμις in mathematical contexts is standardly used of squares, not of their sides. The single exception—and if that is what it is, it is again associated with Theaetetus and the younger Socrates—is Plato's *Politicus* 266ab, where the pair are the beneficiaries of a ponderous mathematical joke having to do with the specific difference between men and pigs, as follows. It is man's nature to walk by the power of two feet, and this is punningly represented as the diagonal of a unit square, that is, a line 2 feet in square/power (δυνάμει δίπους, $\sqrt{2}$ feet); a pig, by contrast, walks by the power of four feet, and a line 4 feet in square/power ($\sqrt{4}$ feet) is the diagonal of (the square on) the first diagonal or, as it is also described (266b 5–6), *the diagonal of our δύναμις*.[12] If this last phrase is taken, as with some translators,[13] as equivalent to "the diagonal of the diagonal" at 266a 9–10, then δύναμις refers again to the incommensurable diagonal $\sqrt{2}$, not to the 2-foot square upon it; but a reference to the square itself[14] would serve as well, or better, for the joke to work. Either way, it is natural to suppose that the *Theaetetus* terminology may be an idiosyncrasy of the young mathematicians' own work from the early days of the theory of irrationals, before vocabulary had crystallized. If so, its inclusion would be part of Plato's tribute to their contribution.

That said, it is to be remarked that when earlier I paraphrased δύναμις as "square line" (in contrast to "length line"), this was calculated to bring out the point that the new *application* which the term receives in the boys' definition does not cancel, but depends upon, its standard meaning—"square." Immense heat has been generated over the terminology of the passage through failure to make any distinction between meaning and application. The cause of the trouble is that while δύναμις is applied to incommensurable lines in the definition at 148b 1, it is also, as already noted, used earlier to specify what Theodorus' demonstrations were about, so it has been a matter of controversy whether at that earlier stage the term stands for the *sides* of a series of squares or for the *squares* themselves, both of which were involved in the exercise.

[12]Diagram and elucidation in J. B. Skemp, *Plato's Statesman* (London: Routledge & Kegan Paul, 1952), p. 139, n. 1; Szabó, *Anfänge*, pp. 90–93.

[13]E.g., Skemp, *Plato's Statesman*, p. 139; A. E. Taylor, *Plato: The Sophist and The Statesman* (London: Nelson, 1961), p. 269.

[14]As in the translation of Otto Apelt, *Platons Dialog Politikos* (Leipzig: Felix Meiner, 1914), p. 35; also Szabó, *loc. cit.*

The latter view was taken by Sachs, following Vogt, and is defended at length by Szabó,[15] but the former has found wider support, being favored by Heath, von Fritz, van der Waerden,[16] and by the greater number of editors and translators since Heindorf.[17] (McDowell's translation is nicely, and perhaps deliberately, ambiguous between the two.) This majority opinion aims to avoid what would, it is felt, be an intolerable shift of meaning (from "square" to "incommensurable side"), but has to concede a narrowing of meaning (from "side" to "incommensurable side") and adduces no parallel for δύναμις in the sense of "side, whether commensurable or incommensurable." It incurs, in addition, a grammatical objection. If δύναμις meant "side," the phrase περὶ δυνάμεων . . . τῆς τε τρίποδος πέρι καὶ πεντέποδος at 147d 3–4 ought to mean "concerning sides 3 foot and 5 foot *in length,*" whereas the sense requires "3 foot and 5 foot in square," for which construction with δύναμις no one has been able to supply a satisfactory parallel or explanation.[18] In any case, the difficulty these scholars aim to avoid is illusory: δύναμις is *applied* to incommensurable lines without *meaning* "line" or "incommensurable line." Given what we have yet to confirm, that the word means "square," it is precisely in virtue of this meaning that it can be adapted to serve as a name for incommensurable lines: in its naming function it alludes to the fact that the lines in question are commensurable in square but not in length, just as the lines which are commensurable in length as well as in square are termed "lengths" or "length lines."

Ancient scholars made the reverse mistake. Evidently they could not conceive how a word meaning "square" might be applied to something that was not actually a square. An anonymous commentary on Plato's *Theaetetus,* which has survived on papyrus from the second century A.D.,[19] observes that "the ancients called squares δυνάμεις" (27, 31–3) and goes on to construe μῆκος and δύναμις in the boys' definition as themselves denominating two species of square (26, 26–48; 33, 8–16; 40, 39–41; 45, 10–14), in plain defiance of the Platonic text (as threatens to show at 41, 8–16).[20] And

[15]Sachs, *De Theaeteto,* pp. 45–46; Vogt, "Entdeckungsgeschichte," pp. 99, 113–114; Szabó, "Der mathematische Begriff," *passim,* and *Anfänge,* pp. 15 ff., 43 ff.; also Heller, "Ein Beitrag," pp. 13, 52; and, in the last century, George Johnston Allman, *Greek Geometry from Thales to Euclid* (Dublin: Hodges, Figgis; London: Longmans, Green, 1889), p. 208 with n. 5.

[16]Heath, *Greek Mathematics,* Vol. I, pp. 203–204, 209, n. 2; von Fritz, "Theaitetos," p. 1354, and "Theodoros," p. 1815; van der Waerden, "Die Arithmetik," p. 249, and *Science Awakening,* p. 166, although in "Nachtrag 1963" to his "Die Arithmetik" (Becker, *Zur Geschichte der griechischen Mathematik,* p. 254) he switches allegiance to the other view in response to the arguments of Szabó, "Der mathematische Begriff"; see also James Gow, *A Short History of Greek Mathematics* (Cambridge: Cambridge University Press, 1884), p. 78, n. 1.

[17]L. F. Heindorf, *Platonis dialogi selecti,* Vol. II (Berlin: Nauck, 1805), p. 300. Szabó, "Der mathematische Begriff," p. 225, incorrectly assigns the origin of this view to Paul Tannery, "Le nombre nuptial dans Platon," *Revue Philosophique de la France et de l'Étranger,* 1876, *1*:185, n. 1. In a later paper, "Sur la langue mathématique de Platon," *Annales de la Faculté des Lettres de Bordeaux,* 1884, N.S. *1* (No. 3):95–100, and even more emphatically in "L'hypothèse géométrique du Ménon de Platon," *Archiv für Geschichte der Philosophie,* 1889, *2*:511, Tannery proposed that throughout our passage δυναμένη should be read for δύναμις to secure an unproblematic reference to lines rather than squares. (These references can also be found in Vols. I and II of Paul Tannery, *Mémoires scientifiques,* ed. J. L. Heiberg and H. G. Zeuthen (Toulouse: Édouard Privat; Paris: Gauthier-Villars, 1912), Vol. I, p. 11, n. 2; Vol. II, pp. 91–98, 402, respectively.) Not surprisingly, such drastic surgery was universally rejected.

[18]The objection is rightly urged by Szabó, "Der mathematische Begriff," pp. 226–227. The attempted solutions of older editors are examined and rebutted by Thomson in William Thomson and Gustav Junge, *The Commentary of Pappus on Book X of Euclid's Elements* (Cambridge, Mass.: Harvard University Press, 1930), App. A, pp. 180–181.

[19]H. Diels and W. Schubart, eds., *Anonymer Kommentar zu Platons Theaetet,* Berliner Klassikertexte (Berlin: Weidmann, 1905).

[20]The commentator's construal is perfectly clear, contrary to the attempt of Mansfeld, "Notes," pp. 112–113, to read him as reporting correctly Plato's double use of δύναμις.

the same (impossible) reading of the definition was reproduced by Pappus *circa* 300 A.D. when he discussed the *Theaetetus* passage in the course of his commentary on Book X of Euclid's *Elements*. This commentary survives only in an Arabic translation, and the English version of the relevant portion[21] renders the boys' definition as a classification of sides rather than squares; but a note by the translator[22] tells us that at this point he has assumed that Pappus' original would have contained an accurate report of Plato's text—the Arabic rather suggests a classification of squares—and a summary of Theaetetus' result a little earlier in the commentary is clear evidence that Pappus did indeed understand it as a classification of squares.[23]

This ancient testimony, mistaken as it is about the defined use of δύναμις at the end of the passage, is for that very reason strong grounds for accepting that δύναμις at 147d means "square." We can add direct statements identifying δύναμις in mathematical usage as "square" by Diophantus, *Arithmetica* 4, 14–15 Tannery, and Iamblichus, *In Nicomachi arithmeticam introductionem* 82, 6–7 Pistelli. Later Greek scholars plainly had no inkling that at an earlier period the word might have meant something different, and in paraphrasing 147d they automatically take δύναμις as "square": see the anonymous commentator 25, 40 ff., Pappus 73–74, also Iamblichus, *Theologoumena arithmeticae* 11, 11–16 de Falco. The correctness of their assumption can be confirmed from Greek mathematical usage itself. It is true that in mathematical contexts δύναμις is most frequently (in Euclid, exclusively) found in complex phrases such as (to repeat those already cited) δυνάμει σύμμετρος ("commensurable in square/power"), δυνάμει δίπους ("2 feet in square/power"), where δυνάμει could in theory be construed in the Aristotelian sense "potentially," as indicating that the line in question is able (has the power) to form a certain square.[24] But these phrases are no less easily construed with δυνάμει = "in square," and the matter is clinched, so far as the earlier period is concerned, by Plato, *Timaeus* 31c, where δύναμις signifies a "square" number in contrast to ὄγκος, a "cube" number.[25]

We must settle, then, for δύναμις at 147d meaning "square." That answers the first of the questions with which this section began; an answer to the second, concerning the specially defined use of δύναμις at the end of Theaetetus' story, has already been proposed. As it happens, the conclusions at which we have arrived are identical with the answers given to the same questions by Szabó, who is the one commentator to have grasped the essential point that δύναμις is applied to incommensurable lines *in virtue of* its meaning "square." But his route to these conclusions differs importantly from ours. To fix the meaning of δύναμις he relies less on parallels of usage, and not at all on the testimony of ancient scholarship, but mainly on an elaborate and speculative reconstruction of how the word could have come to mean "square."

[21]Thomson and Junge, *Commentary of Pappus,* p. 74.

[22]*Ibid.*, p. 103, n. 80.

[23]*Ibid.*, p. 73; see also Thomson's App. A, pp. 180 ff., on the term "power." Further confirmation is available in a medieval Latin translation of the commentary, made from a version of the Arabic which was not in all respects identical to the one that survived: Gustav Junge, "Das Fragment der lateinischen Übersetzung des Pappus—Kommentars zum 10. Buche Euklids," *Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik,* 1936, *3*:17, lines 13–15, where *illa* in the rendering of the classification at the place which Thomson amended can only stand for *virtus* = δύναμις, i.e., square.

[24]Cf. Karl Bärthlein, "Über das Verhältnis der Aristoteles zur Dynamislehre der griechischen Mathematiker," *Rheinisches Museum für Philologie,* 1965, *108*:35–61; *contra,* Szabó, *Anfänge,* pp. 19–22, 44, n. 14.

[25]Cited long ago by Sachs, *De Theaeteto,* p. 45, n. 2, together with *Politicus* 266b, which we discussed above, and *Timaeus* 54b, which is, however, another complex phrase κατὰ δύναμιν. *Republic* 587d, cited by Szabó, "Der mathematische Begriff," pp. 221–223, is more help, for there κατὰ δύναμιν contrasts with κατὰ τρίτην αὔξην as "in square" with "in cube." For later mathematical usage it is sufficient to take a glance at the *Index verborum* in standard editions of Archimedes, Pappus, or Diophantus.

By his account, this application of the term is a metaphor from its financial sense of "worth, value," developed in connection with the operation of "squaring"[26] a given rectangle by finding the mean proportional between its sides. The square on this proportional is equal in area to the rectangle (see Euclid, *Elements* II 14, VI 13 and 17), and can be called δύναμις because it represents the "square-value" (*Quadratwert*) of the latter, what it is "worth" (ὅσον δύναται) in square. So δύναμις means "square-value of a rectangle," hence "square."[27]

That is step 1 in Szabó's account. Step 2 is a chronological hypothesis. There is reason to believe that the geometrical construction of the mean proportional was already known to Hippocrates of Chios in the second half of the fifth century, for it is presupposed by his famous quadrature of lunes.[28] What is more, Szabó thinks that use of the construction would inevitably lead to reflection on cases where the mean proportional has no whole number expression, thereby to the concept of linear incommensurability; and it is precisely in these cases, for the squares with sides not expressible as whole numbers, that the special term δύναμις would be needed.[29] Hence a knowledge of incommensurability is bound up with the δύναμις-terminology and both must predate Hippocrates; in the well-known Eudemus-Simplicius report of Hippocrates' quadrature of lunes, a statement that similar segments of circles have the same ratio to one another as their bases have δυνάμει, in square (Simplicius, *In Aristotelis Physica* 61, 6–7 Diels; cf. 61, 9; 62, 18 *et al.*), could well represent the fifth-century mathematician's own use of the terminology.[30] Thus the material Theodorus taught in 399 had long been familiar to mathematicians and is not to be acclaimed as his own achievement.[31]

So much for Theodorus. Step 3, the debunking of Theaetetus, follows swiftly after. But before following further ourselves we should pause to reflect on some of the inferences we have been invited to draw. Suppose, first, we accept step 1. It is not at all clear how rapidly reflection on the cases Szabó mentions would reach anything like *proofs* of incommensurability,[32] but in any event the most the argument establishes is that these developments were known to Hippocrates by the time he did his quadrature of lunes. Hippocrates and Theodorus are coupled as contemporaries in the only comparative chronological record extant,[33] so for all that has been shown to

[26] τετραγωνίζειν, used here at 148a 7.

[27] Szabó, *Anfänge,* pp. 43–60.

[28] Here Szabó, *Anfänge,* pp. 62, 66, n. 37, can cite the agreement of van der Waerden, "Die Arithmetik," p. 225, n. 28, *Science Awakening,* p. 134.

[29] Szabó's further projected specification of these cases is in terms borrowed from Bk. VIII of Euclid's *Elements,* where propositions 18 and 20 state, in effect, that the mean proportional is a (whole) number if, and only if, the sides are similar plane numbers, i.e., numbers whose factors are proportional (*Elements* VII def. 21). This involves him in a controversial early dating of Bk. VIII. Compare van der Waerden, "Die Arithmetik," pp. 222–226, *Science Awakening,* p. 153, who attributes the work to Archytas on considerations most of which (unlike the counter-considerations of Szabó, *Anfänge,* pp. 98–100, 229 ff.) presuppose no view as to when and how the theory of irrationals first developed; van der Waerden's attribution is approved by Kurt von Fritz, "Archytas of Tarentum" in *Dictionary of Scientific Biography,* Vol. I (New York: Scribner's, 1970), pp. 231–232.

[30] That the terminology reflects Hippocrates' own language is also suggested by Thomas L. Heath, *Mathematics in Aristotle* (Oxford: Clarendon Press, 1949), pp. 207–208. If so, this would be the earliest attested mathematical occurrence of δύναμις.

[31] Szabó, *Anfänge,* pp. 55–80.

[32] In a later section of the book, Szabó (*ibid.*, pp. 238–239; cf. 264) concedes as much himself when he notes that to have reached *Elements* VIII 18 and 20 is not yet to have demonstrated that there do exist incommensurable lines; cf. Szabó, "Theaitetos," pp. 345 ff.; *Anfänge,* pp. 263–287.

[33] Eudemus *apud* Proclus, *In Euclidis Elementa I* 66, 4–7 Friedlein; Iamblichus, *De communi mathematica scientia* 77, 25–78, 1 Festa.

the contrary, what Hippocrates knew could have been the work of Theodorus. (That the relevant concepts and operations were not new in a fictitious lesson set in 399 and written around 369 is, of course, not to the point, although Szabó sometimes speaks as if it were.) Indeed, Szabó's hypothetical reconstruction of the history of the δύναμις-terminology could perfectly well lead us to ascribe a major role in the events to Theodorus, precisely on the strength of the Platonic passage under dispute. Everything Szabó gets out of the early part of Theaetetus' story Plato could have put in, just as historians of mathematics have traditionally supposed, in order to honor the elder man's achievement.

But, of course, the reconstruction at step 1, ingenious though it is, remains a hypothetical speculation. There is more than one way in which δύναμις could have come to be applied to squares. For example, when Aristotle in the *Metaphysics* (1019b 33–4, 1046a 6–8) remarks that δυνάμεις in geometry are a different sort of thing from other δυνάμεις (potentialities, powers), being homonymously or metaphorically so called on account of a resemblance to these, the commentator Alexander of Aphrodisias (*In Aristotelis Metaphysica* 394, 34–6 Hayduck) explains that a square is called δύναμις because it is ὃ δύναται ἡ πλευρά, what the side is able (to produce).[34] We are really in no position to improve upon this simple and straightforward derivation. That is why it was important to establish that Theodorus' δυνάμεις are squares on grounds having nothing to do with the genesis of the concept. That they are squares is both more secure than any genetic hypothesis can claim to be and sufficient to allow exegesis to proceed.

To pass, then, to Theaetetus' own part in the story and to step 3 in Szabó's account.[35] Szabó contends that μῆκος and δύναμις ("length" and "power") in the boys' definition at 148ab are nothing but abbreviations of the technical phrases (familiar in Euclid) μήκει σύμμετρος and δυνάμει σύμμετρος ("commensurable in length," "commensurable in square"); this allows him to insist that the concepts cannot have been Theaetetus' creation.[36] I was not so specific as to how δύναμις would be applied to incommensurable lines in virtue of its meaning "square," but the route through abbreviation is likely enough.[37] Less plausibly, however, Szabó finds that by comparison with the fuller technical descriptions, which are precise and clear, the abbreviations are imprecise and misleading, a sign of student immaturity.[38] He is entitled to his opinion, but what has to be shown is that this was Plato's opinion. It is

[34]Similarly the anonymous commentator, 27, 31 ff. This mathematical use of the verb δύνασθαι is to be noted in our passage of the *Theaetetus* at 148b 2: δύνασθαι ἐπίπεδον. See also Plato, *Republic* 546b; Aristotle, *De incessu animalium* 709a 1 and 19; Euclid, *Elements* X def. 4, X 21, *et al*.; the Hippocrates report of Eudemus *apud* Simplicius, *In Aristotelis Physica* 63, 10–13 *et al*.; Proclus, *In Euclidis Elementa I* 8, 12–14 Friedlein, *In Platonis Republicam* II 36, 9–12 Kroll (these two in close connection with δύναμις); Archimedes, *passim* (J. L. Heiberg, *Archimedis opera omnia*, Leipzig: Teubner, 1915, Index I *s.v.*). Neither Alexander nor the other writers show any trace or awareness of the financial metaphor by which Szabó, *Anfänge*, pp. 45–47, would explain the construction. Szabó, "Der mathematische Begriff," pp. 244–247 (cf. *Anfänge*, pp. 45–46), concedes that the examples are not exclusively, or even mainly, from contexts dealing with the squaring of a rectangle, but he argues that the mathematical use of the verb would have originated from such contexts because the "square-value" of other rectilinear plane figures was determined by first constructing an equivalent rectangle and then squaring it (Euclid, *Elements* II, 14). What he does not seem to appreciate (*Anfänge*, p. 52, n. 28) is that while his financial metaphor supplies an explanation of δύνασθαι followed by a quantitative or measuring expression (δύνασθαι ἴσον, διπλάσιον, etc.), it is less easy to derive from this the construction δύνασθαι ἐπίπεδον, τετράγωνον, which Alexander takes as primary, than it is to proceed the other way round.

[35]Szabó, *Anfänge*, pp. 80–100.

[36]*Contra* e.g. van der Waerden, *Science Awakening*, p. 168.

[37]The suggestion in fact goes back at least to Vogt, "Entdeckungsgeschichte," p. 114.

[38]Cf. also *ibid*., pp. 114, 127.

Plato's intentions we have to read, but there is no hint in the text of any such adverse judgment on his part.[39]

Nevertheless, Szabó proceeds to argue that all the two boys have done is hand back to Theodorus what he doubtless hoped to elicit, a classification of squares (*sic*), which was presupposed already by the concepts used in the lesson: Theodorus after all was sorting squares according as they have commensurable or incommensurable sides, and it would be obvious all along that whole number squares with incommensurable sides still have commensurable areas. No doubt it would be, but, once again, what has to be shown is that this was Plato's verdict. In Plato's text what is depicted as the boys' central achievement is not the recognition that the squares with incommensurable sides have commensurable areas—for all that is said to the contrary this could have been part of Theodorus' instruction, as indeed the anonymous commentator (28, 18–34) imagines it was—but their finding a general answer to the question *which* lines have the property of being commensurable in square only.[40] On this crucial point Szabó thinks it sufficient to allege that the answer has to be "reconstructed" from the boys' classification, which is not given in the form of a mathematical proposition but as a definition or designation of two classes of line.[41] At best this is a quibble over the wording of what, it should be remembered, is informal narrative, not a mathematical treatise. Certainly Theodorus in his lesson had shown that some squares are such as to be commensurable in square only, but he gave no general condition for the property. His pupils did. A slight infelicity (if such it be) in Theaetetus' terminology (however this was arrived at) is quite inadequate justification for Szabó's dismissive estimate of their accomplishment.

What is true—and this helps to explain both why ancient scholars found it so difficult to read what Plato actually wrote at 148ab and why modern scholars have been tempted to the idea that δύναμις initially means "side"—is that if δύναμις means "square" at 147d 3, it means the same at 147d 8–e 1, where Theaetetus speaks of deciding to look for a general characterization of the δυνάμεις which formed the subject of Theodorus' incommensurability proofs. We seem, then, to be promised a definition pertaining to squares, whereas the outcome is a classification of lines. How serious is this discrepancy?

The key sentence is 147d 7–e 1, which I render as follows:

> Since the δυνάμεις were turning out[42] to be unlimited in number, it occurred to us to attempt to collect them up into a single way of speaking [i.e., a formula or definition] of all these δυνάμεις together.

Theaetetus is recounting the thoughts suggested to himself and his companion by and during Theodorus' lesson, and the idea that there is an endless series of whole number squares (or sides of such squares) would hardly need to be prompted by a process as protracted as Theodorus' lesson.[43] That there are an indefinite, perhaps infinite, number of squares with incommensurable sides, on the other hand, is precisely the hypothesis that would suggest itself as Theodorus proceeded from case to case proving more and yet more examples of incommensurability, perhaps by a method

[39] Nor with respect to a couple of lesser terminological inexactitudes detected by Szabó, *Anfänge*, p. 83 (cf. also pp. 108–109).
[40] See van der Waerden, *Science Awakening*, p. 168.
[41] Szabó, *Anfänge*, pp. 97–98.
[42] Note the imperfect ἐφαίνοντο.
[43] *Pace* Szabó, "Der mathematische Begriff," pp. 232–233.

which could be endlessly reapplied.[44] Therefore, it is likely that, in context, "all these δυνάμεις" refers to squares with incommensurable sides rather than to squares generally.[45] But even if this point of detail is not accepted, no change of *mathematical* substance is involved when the definition is eventually given in terms of lines. Whether incommensurability is taken as a property that certain squares have in respect of their sides or as a property of the sides themselves is just a matter of which way one chooses to express the same facts. (That is why a competent mathematician like Pappus could recast the definition as a classification of squares.) Such linguistic variation is of little account when it is perfectly plain that the interest and importance of the boys' endeavor lies in their formulation of a general condition for (linear) incommensurability. It is this, and this alone, that Socrates subsequently holds up as a model for Theaetetus to follow in answering the question "What is knowledge?" (148d).

THIRD PROBLEM: THE BEGINNING AND ENDING OF THEODORUS' LESSON

Theodorus, it will be recalled, began his demonstrations on a 3-foot square and ended on one of 17 square feet. Already in ancient times there was extensive debate about the reasons for these termini.

To take first the question why the side of a 2-foot square is not listed among those proved incommensurable by Theodorus. The anonymous commentary (28, 37–29, 40) recounts three rival explanations: (1) that Plato (!) had already dealt with it in the *Meno* (84d–85b), (2) that the 2-foot square is in fact implicitly included since it can be divided into equals (*sc.* equal areas, which, as the commentator objects, is entirely beside the point), (3) that to construct a square on the diagonal for the case of $\sqrt{2}$ is no trouble, whereas to prove subsequent incommensurabilities a construction of some complexity is required. The commentator backs (3), which may be his own contribution, and he outlines a construction for the purpose like that of *Elements* II 14, but in this and in his further explanations he is clearly guessing; thus at 44, 26–40 he reveals that he has no idea what terminology Theaetetus used when extending his definition to solids. The importance of his testimony is what it shows about the state of knowledge of a reasonably conscientious scholar in the second century A.D. He knows other people's discussion of the question, but neither through them nor on his own does he have access to genuine historical information (beyond the dialogue itself) on Theodorus' work in the area of irrationals.

This should not necessarily dispose us to share Szabó's skepticism about Theodorus' contribution.[46] In later antiquity data on earlier Greek mathematics derived largely from the history of the subject written by the Peripatetic Eudemus in the second half of the fourth century B.C. What information was available would depend very much on Eudemus' selection of relevant material. Notices of Theodorus outside our dialogue do no more than remark on his distinction in general terms, which

[44]So Vogt, "Entdeckungsgeschichte," pp. 106–107, but see below.

[45]For what it is worth, the anonymous commentator 26, 6–8, agrees. Note also, with Vogt, "Entdeckungsgeschichte," p. 101, that 147d 4 implies that squares with commensurable sides were omitted from the sequence.

[46]Less radical doubts about how far Plato means to attribute to Theodorus an important new discovery are expressed by Allman, *Greek Geometry*, p. 213; Heath, *Greek Mathematics*, Vol. I, p. 205 (inconsistently with p. 155); A. Wasserstein, "Theaetetus and the History of the Theory of Numbers," *Classical Quarterly*, 1958, N.S. *8*:166, 171.

confirms only that there was substance to his reputation.[47] The picture would have been clearer, of course, to the dialogue's first audience, which was contemporary with Theaetetus and his work, but at least in regard to the question why nothing is said about $\sqrt{2}$, later readers were perhaps hardly better off than ourselves.

Nevertheless, the commentator and his like saw that the question needs to be asked, as that first audience must have asked it. The fact that Plato has Theodorus leave out what was by far the most celebrated example of incommensurability seems a rather convincing sign of an intention to demarcate Theodorus' own contribution. Accordingly, modern authorities make the inference that Theodorus did not undertake to prove the irrationality of $\sqrt{2}$ because it was an old and familiar (traditionally, Pythagorean) result.[48] One can only agree that this is the most probable explanation.

At the other end of the lesson matters are less straightforward. As before, the anonymous commentator (34, 32 ff.) raises the question why Theodorus stopped at the square of 17 square feet and retails various answers. One suggestion obscurely drags in Theodorus' interests in the theory of music,[49] taking a hint from the list of his professional concerns at *Theaetetus* 145a: geometry, astronomy, arithmetic, and music. Another understands Plato to mean there was no particular reason for Theodorus' stopping where he did. The commentator himself offers an improvement having to do with the mathematically irrelevant fact that the 16-foot square is the only one where the length of the perimeter corresponds (numerically) to the area enclosed ($4 + 4 + 4 + 4 = 4 \times 4$), while the 17-foot square is the first with an area greater (numerically) than the perimeter.[50] Again as before, these views carry no authority, but we can profit from them nonetheless. In particular, we can learn what a Greek ear made of the sentence at 147d 6, *ἐν δὲ ταύτῃ πως ἐνέσχετο*, which has been thought to hide a clue about the method used in Theodorus' incommensurability proofs.

The sentence could mean three things: (a) "at that point for some reason [*sc.* for no particular reason that Theaetetus knows of] he stopped," (b) "at that point for some reason [*sc.* for some particular reason] he stopped," (c) "at that point he somehow got

[47] Plato, *Politicus* 257a; Xenophon, *Memorabilia* 4.2.10; Iamblichus, *De communi mathematica scientia* 77, 24–78, 1 Festa; Proclus, *In Euclidis Elementa I* 66, 6–7 Friedlein. Proclus, *ibid.* 118, 7–8, on a question about curves, is a passage cited by von Fritz, "Theodoros," p. 1812, and van der Waerden, *Science Awakening,* p. 146, as an indication of Theodorus having other interests besides irrationals, but it may refer to a different Theodorus; see Glenn R. Morrow, *Proclus: A Commentary on the First Book of Euclid's Elements* (Princeton: Princeton University Press, 1970), p. 95, n. 70. Perhaps mention should also be made of one other context in which Theodorus' name occurs: the story that Plato's travels after the death of Socrates included, besides a stay at Megara, a visit to Theodorus in his home town of Cyrene (Hermodorus *apud* Diogenes Laertius III 6). There have been varying evaluations of the scholarly credentials of this story (see W. K. C. Guthrie, *A History of Greek Philosophy,* Vol. IV, Cambridge: Cambridge University Press, 1975, pp. 14–16), but if it is true, a compliment to Theodorus would fit well in a dialogue which begins by paying respect to Plato's Megarian hosts.

[48] Vogt, "Entdeckungsgeschichte," p. 111; Sachs, *De Theaeteto,* pp. 49–52; Heath, *Greek Mathematics,* Vol. I, pp. 155–157; von Fritz, "Theodorus," p. 1813; van der Waerden, "Die Arithmetik," pp. 228–229, *Science Awakening,* p. 110; Wasserstein, "Theaetetus," pp. 165–166; Burkert, *Lore and Science,* p. 463; exceptionally, Frajese, "Perchè Teodoro," pp. 426 ff., offers a different account. Szabó, *Anfänge,* pp. 40–43, 76, is naturally opposed to the traditional view, but he says nothing to meet the point (clearly stated in Vogt, "Entdeckungsgeschichte," p. 111; von Fritz, "Theodoros," pp. 1812–1813; von Fritz, "Discovery," p. 385) that the omission of $\sqrt{2}$ shows something about Plato's intentions.

[49] For an attempt at clarification, see Wasserstein, "Theaetetus," pp. 172 ff.

[50] Similarly Iamblichus, *Theologoumena arithmeticae* 11, 11–16 de Falco, also 29, 6–10; and E. Stamatis, "Ἐπὶ τοῦ μαθηματικοῦ χωρίου τοῦ Θεαιτήτου τοῦ Πλατῶνος," Πρακτικὰ τῆς Ἀκαδημίας Ἀθηνῶν, 1956, *31*:10–16, who also develops connections with the theory of music. At 44, 1–20, the anonymous commentator has an analogous idea about cubes.

tied up" (McDowell). The anonymous commentator saw the issue as one between (a) and (b) (see 35, 13–21), and so have historians of mathematics in modern times. From reading (a) Vogt inferred that if Theodorus had no particular reason for stopping at $\sqrt{17}$, his method of proof must have been an endlessly reapplicable one: specifically, an adaptation of the classic indirect proof by which it was shown that the diagonal of a square ($\sqrt{2}$) must be incommensurable with its side (unity), otherwise the same number will be both odd and even (see Aristotle, *Prior Analytics* 41a 26–31, 50a 37–8; Euclid, *Elements* X, App. 27).[51] What is more, on the basis that the proof was of a kind to be transferable from each case to the next without end, Vogt further argued that it is really Theodorus who should be credited with discovering a general law for linear incommensurability.[52]

If these seem large consequences to draw from a single ambiguous sentence, reading (b) has supported the postulation of a mathematical reason for terminating at $\sqrt{17}$. From the classic papers of H. G. Zeuthen onward, much ingenuity has gone into the search for a method of proof which would result in special difficulties at or after $\sqrt{17}$.[53] There is no need to discuss the various suggestions in detail here. The point to be made is that there was no clear textual warrant for preferring a proof of this character until in 1957 R. Hackforth pointed out that although for the verb ἐνέσχετο in the disputed sentence the lexicon gives the sense "came to a standstill," this is the only place cited for such a meaning.[54] Normally the verb would mean "be held up, entangled" = reading (c). And if, as Hackforth argues and McDowell accepts in his translation, that is its sense in the present passage, the case for a specifically mathematical block at $\sqrt{17}$ becomes very strong indeed.[55]

Yet we see from the anonymous commentator that the philological argument is not decisive. It is not only he that glosses ἐνέσχετο without hesitation as "stopped" (34, 35: ἔστη; 35, 16 and 21: στῆναι), so also did the predecessor whom he reports as favoring reading (a) and so too did Iamblichus (*Theologoumena arithmeticae* 11, 14–15: παύεσθαί πως).[56] And the commentator's argument for (b) against this reading looks to be on the basis that πως, not that ἐνέσχετο, implies that Theodorus had a reason for stopping where he did (35, 17–21): the commentator states that the words πως ἐνέσχετο show there was such a reason, and while this is not perfectly clear, the conjectural reason he goes on to supply, concerning the numerical relations between area and perimeter (see above), carries no suggestion of entanglement or

[51] Vogt, "Entdeckungsgeschichte," pp. 105–111; similarly von Fritz, "Theodoros," p. 1824; Wasserstein, "Theaetetus," pp. 165, 171. A different version of reading (a) is that of Szabo, *Anfänge*, p. 110; he supposes that for pedagogical reasons Theodorus pretended to be unable to continue further, leaving his pupils to puzzle out the way.

[52] Vogt, "Entdeckungsgeschichte," pp. 109–110.

[53] H. G. Zeuthen, "Sur la constitution des livres arithmétiques des Éléments d'Euclide et leur rapport à la question de l'irrationalité," *Oversigt over det Kongelige Danske Videnskabernes Selskabs Fordhandlinger*, 1910, 395–435; "Sur les connaissances géométriques des grecs avant la reforme platonicienne de la géométrie," *ibid.*, 1913, 431–474; "Sur l'origine historique de la connaissance des quantités irrationelles," *ibid.* 1915, 333–362. Cf. van der Waerden, "Die Arithmetik," pp. 249–254; *contra*, von Fritz, "Theodoros," pp. 1817–1825, also the comprehensive review of Zeuthen's solution and its successors in Heller, "Ein Beitrag."

[54] R. Hackforth, "Notes on Plato's *Theaetetus*," *Mnemosyne*, 1957, Ser. 4, *10*:128, referring to H. G. Liddell, R. Scott, H. S. Jones, *A Greek-English Lexicon* (9th ed., Oxford: Clarendon Press, 1940), *s.v.*

[55] Thus Hackforth's note is used as clinching evidence for Zeuthen's general scheme of interpretation by Malcolm S. Brown, "*Theaetetus:* Knowledge as Continued Learning," *Journal of the History of Philosophy*, 1969, *7*:367, who goes on to elaborate his own version of that scheme.

[56] Their view of the term is defended, against Hackforth, by Mansfeld, "Notes," pp. 113–114, with the observation that the lexicon instances for "entangled" depend on a contextual understanding of the difficulties concerned; the verb itself is neutral.

difficulty. There is no sign that Hackforth's account of the verb, and reading (c) which it supports, was a live option in ancient controversy.

In the face of this impasse the question that needs to be asked is whether *Plato* has any reason to leave a hint, let alone so indeterminate and ambiguous a hint, as to the mathematical methods used in Theodorus' lesson. As every reader of the dialogue knows, the mathematical scene illustrates a point about definition and examples. When Theaetetus is first asked what knowledge is, he replies by giving examples of knowledge: geometry and the other mathematical sciences he is learning with Theodorus, cobblery and other crafts—each and all of these are knowledge (146cd). Socrates puts him right with an analogy: his answer is like that of someone who, on being asked what clay is, replies, "There is potters' clay, brickmakers' clay, and so on, each and all of which are clay," giving a list of clays instead of the simple, straightforward answer, "It is earth mixed with liquid" (146d–147c). It is at this point that Theaetetus says, "It looks easy now, Socrates, when you put it like that" (147c), and proceeds to tell his story. Theodorus' part in the story does not depend on whether or not he could continue past $\sqrt{17}$; his role is to provide *examples* of incommensurability. His case-by-case proof of their incommensurability is mentioned[57] because, if one is not going by a general definition or rule of the kind the boys devised, it is only via construction and proof that examples of incommensurability are forthcoming: construction to obtain a length such as $\sqrt{3}$, which is not marked on any ruler, and proof to show that, divide how you will, you can find no unit to measure without remainder both it and a 1-foot line. Beyond that, Plato has no motive to indicate to the reader whether he has in mind any definite method of proof or any particular cause for Theodorus to stop at $\sqrt{17}$.

This is not to deny, of course, the legitimacy of speculating about what methods would be available to Theodorus or other fifth-century mathematicians for proving various cases of incommensurability. But this must be an independent inquiry; there is no good reason to expect that the answer is to be squeezed out of one ambiguous sentence in Plato's dialogue.

FOURTH PROBLEM: THEAETETUS AND THE TENTH BOOK OF THE ELEMENTS

Euclid, *Elements* X 9 states that two lines are commensurable if the squares upon them have to one another the ratio of a square number to a square number, and conversely; that is, given lines A and B, $A : B = n : m$, where n and m are positive integers, if and only if $A^2 : B^2 = n^2 : m^2$. A scholium to this theorem says that it is alluded to in Plato's *Theaetetus,* only in less general form, and that it is Theaetetus'

[57]It is disputed whether ἔγραφε at 147d 3 connotes the actual theorem-proving (so Heath, *Greek Mathematics,* Vol. I, p. 203, n. 2; *contra,* Szabó, "Der mathematische Begriff," p. 228, *Anfänge,* p. 76), or construction to prove the lines' existence (Vogt, "Entdeckungsgeschichte," p. 101; von Fritz, "Theodoros," pp. 1814–1815), or mere diagrammatic illustration (Szabó, "Der mathematische Begriff," pp. 224–225 [but cf. *Anfänge,* p. 48]; McDowell's translation). If ἔγραφε does not mean "was proving," it is left to ἀποφαίνων at 147d 4 to convey the idea of proving or showing. Not that doubts have not been raised about ἀποφαίνων, too, as to whether it signifies proving or a less formal procedure for showing, making evident (see Szabó, "Der mathematische Begriff," pp. 231–232 [partially withdrawn, "Theaitetos," pp. 323–325, *Anfänge,* p. 76]; Heller, "Ein Beitrag," pp. 34–35, with further references), and as to its presence in the text at all: Burnet omits it from the Oxford text (*Platonis opera* I, 2nd ed.; Oxford: Clarendon Press, 1905), following one of the MSS, although there would seem to be little reason for this decision (most modern authorities retain the word) unless ἔγραφε does mean "was proving," so that ἀποφαίνων is slightly superfluous. The anonymous commentator had ἀποφαίνων in his text and glossed the whole verb complex ἔγραφε . . . ἀποφαίνων by ἐδείκνυεν = was proving (25, 34–35, and 42), which seems the sensible solution.

discovery.[58] Pappus also (72–5), although he is more concerned to bring out the difference between Theaetetus' result and *Elements* X 9, which he credits to Euclid himself, regards the latter as a deliberate generalization of the former, and he refers to others who had taken the same line in terms which suggest that, as in modern times, there had been some controversy on the precise relation between the two.

Now Pappus' treatment of the question is clearly based, so far as concerns Theaetetus' contribution, entirely on Plato's dialogue. This virtually establishes that there was no other evidence to go by and confirms scholarly doubts as to whether the historical basis of the scholium is anything more than a conjecture inspired by comparing dialogue and theorem.[59] At the same time, the conjecture is not an unthinking extrapolation from the dialogue, for the scholiast is as clear as Pappus about the mathematical difference between the two results.

That difference is as follows: the Euclidean theorem explains when two lines stand to one another in a whole number ratio n to m, but in the dialogue Theaetetus gave conditions under which they have the more specific ratio n to 1; the latter is the less general version of which the scholiast speaks. For example (the example used both in the scholium and by Pappus) $\sqrt{18}$ and $\sqrt{8}$, each of which is incommensurable with unity by Theaetetus' definition, are commensurable with each other on the Euclidean criterion, since $18 : 8 = 3^2 : 2^2$.[60] Regarding roots as numbers, only the dialogue yields what for us would often be the significant thing, a determination of the rationality or irrationality of $\sqrt{n}$;[61] but, as already emphasized, the Greeks did not regard what we call $\sqrt{n}$ as a number. It is indicative of the geometrical orientation of Greek mathematical thinking that Pappus should commend the Euclidean theorem for its greater generality, as such an advance on Theaetetus' finding, and leave the matter there.

No one today, however, would follow Pappus in attributing the discovery of *Elements* X 9 to Euclid himself. Like the scholiast, Pappus is convinced that our *Theaetetus* passage records a real contribution to mathematical knowledge, but in the absence of firm historical data he resorts to inference and conjectural reconstruction. His inference is that, because *Elements* X 9 is different from the definition in the dialogue, Theaetetus cannot be credited with both. The scholiast's contrary view is that, although they are different, he can. Our fourth problem is to make up our minds where to stand on this dispute.

[58]Scholium 62 to Euclid, *Elements* X: J. L. Heiberg, *Euclidis Elementa* V (Leipzig: Teubner, 1888), pp. 450–452. There is evidence in another scholium that 62, or part of it, is due to Proclus; see J. L. Heiberg, "Paralipomena zu Euklid," *Hermes,* 1903, *82*:341, 345–346, who remains doubtful, having reason to think that the attribution may be no more than a Byzantine scholar's guess. The attribution to Proclus is accepted on the basis of Heiberg's evidence, but without mention of his reservations, by Sachs, *De Theaeteto,* p. 12, n. 1; Brown, *"Theaetetus,"* p. 362, n. 8.

[59]Vogt, "Entdeckungsgeschichte," p. 115; von Fritz, "Theaitetos," p. 1357; Szabó, "Theaitetos," pp. 336–338, 343; *Anfänge,* pp. 100–104, all of whom turn the doubt in favor of their general scheme of interpretation (which in Vogt's case involves attributing the essentials of X 9 to Theodorus and in Szabó's means that X 9 predates Theodorus). Van der Waerden, "Die Arithmetik," pp. 237–247, is more sanguine, thinking that the scholiast's report may go back to a source acquainted with Theaetetus' own writings—e.g., a commentary on the dialogue (note, however, that there is no sign of such acquaintance in the anonymous commentary which has come down to us).

[60]It is to be remarked that the ancients' grasp of this critical point is not always matched in modern discussions: Szabó, "Theaitetos," pp. 337, 343, *Anfänge,* p. 102, credits dialogue and theorem with "the same classification of squares" (that is how he can argue that *Elements* X 9 must predate Theodorus); Brown, *"Theaetetus,"* p. 370, n. 30, is puzzled to know what difference in generality the scholiast could intend. Nor have I been able to find in the modern literature any acknowledgement of the admirably clear exposition of the matter which both Pappus and the scholiast provide.

[61]See Helmut Hasse and Heinrich Scholz, "Die Grundlagenkrisis der griechischen Mathematik," *Kantstudien,* 1928, *33*:8 n.; von Fritz, "Theaitetos," pp. 1356–1357, 1359.

The problem is deepened by the one really useful item of information which ancient readers did find: evidence that Theaetetus had done work on species of irrational lines more complex than those defined in the *Theaetetus,* the medial, binomial, and apotome, which are incommensurable with a given unit-line in square as well as in length. Specifically, Pappus found it in Eudemus and wrote at the beginning of his commentary (63; see also 138) as follows:

> The aim of Book X of Euclid's treatise on the Elements is to investigate the commensurable and incommensurable, the rational and irrational continuous quantities. This science (or knowledge) had its origin in the sect (or school) of Pythagoras, but underwent an important development at the hands of the Athenian, Theaetetus, who had a natural aptitude for this as for other branches of mathematics most worthy of admiration. One of the most happily endowed of men, he patiently pursued the investigation of the truth contained in these [branches of] science (or knowledge), as Plato bears witness for him in the book which he called after him, and was in my opinion the chief means of establishing exact distinctions and irrefragable proofs with respect to the above-mentioned quantities. For . . . it was . . . Theaetetus who distinguished the *powers* (i.e. the squares) which are commensurable in length, from those which are incommensurable (i.e. in length), and who divided the more generally known irrational lines according to the different means, assigning the medial line to geometry, the binomial to arithmetic, and the apotome to harmony, as is stated by Eudemus, the Peripatetic.[62]

In the last sentence of this excerpt, the clause about Theaetetus distinguishing commensurable and incommensurable powers is probably of no independent worth: Pappus having already indicated that the *Theaetetus* was his starting inspiration, the chances are that this part of his report derives directly from there. The probability can be increased by noticing a certain disconnection between this clause and its sequel: had Eudemus been the author of the whole, one would expect the point to be made that Theaetetus distinguished lines commensurable in square but not in length from those incommensurable in square as well as in length.[63] Eudemus may still have said something to encourage the practice of treating the *Theaetetus* as an historical source, but it is the remainder of the report, entirely independent of the dialogue, which carries his authority. As such it is pure gold.

The medial, binomial, and apotome occupy a central position in the study of irrationals in Book X of the *Elements.* Accordingly, modern historians infer that the theory of irrationals in Book X is in substance the work of Theaetetus.[64] To support the inference they can point, not only to the report from Eudemus, who wrote before Euclid, but also to important connections between Book X and Book XIII of the

[62]Thomson and Junge, *Commentary of Pappus;* translator's parentheses. For an explanation of the reference to different means, see Thomas L. Heath, *The Thirteen Books of Euclid's Elements* (2nd ed.; Cambridge: Cambridge University Press, 1926), Vol. III, p. 4.

[63]This consideration is due to von Fritz, "Theaitetos," p. 1355. Szabó, *Anfänge,* pp. 103–104, reaches the same conclusion with less argument.

[64]Sachs, *De Theaeteto,* pp. 43 ff.; Heath, *Greek Mathematics* I, pp. 211–212; von Fritz, "Theaitetos," pp. 1355 ff.; van der Waerden, "Die Arithmetik," pp. 235 ff.; *Science Awakening,* pp. 168 ff. "In substance" is said advisedly, because Euclid's presentation is at various points (including the proof of X 9) adapted to developments in the theory of proportions later than Theaetetus, who is thought to have worked with an earlier, less general concept of proportionality; see Oskar Becker, "Eudoxos-Studien I: Eine voreudoxische Proportionenlehre und ihre Spuren bei Aristoteles und Euklid," *Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik,* Abt. B: Stud., Bd. 2, 1932–1933, 311–333; von Fritz, "Theaitetos," pp. 1358, 1359–1360, 1362–1363; van der Waerden, "Die Arithmetik," pp. 233, 241–249, *Science Awakening,* pp. 159, 175 ff. (Heath, *Mathematics in Aristotle,* pp. 81–83, has reservations about Becker's reconstruction; Szabó, "Ein Beileg für die voreudoxische Proportionenlehre?," *Archiv für Begriffsgeschichte,* 1964, *9*:151–171, mounts a wholesale attack on it, but there is no doubt about the main point, that a certain amount of adaptation has gone into Bk. X.)

Elements, which incorporates Theaetetus' other known achievement—his work on the five regular polyhedra.[65] Van der Waerden concludes: "The author of Book XIII knew the results of Book X, but . . . moreover, the theory of Book X was developed with a view to its applications in Book XIII. This makes inevitable the conclusion that the two books are due to the same author. We already know his name: Theaetetus."[66]

Once again, let us remind ourselves that the full extent of Theaetetus' mathematical achievements would have been familiar to many in the audience on the occasion of the first reading of the *Theaetetus* in the Academy, an occasion which must in part have had the character of a memorial to a departed colleague. Socrates the Younger, whose early collaboration with Theaetetus is mentioned again at *Sophist* 218b (cf. *Politicus* 266a 6–7), could well have been alive and present in person, now a middle-aged member of the Academy. We know from Plato's *Republic* (528b ff.) and *Timaeus* (53d ff.)[67] how highly Theaetetus' work in solid geometry was valued in the Academy. If the solid geometry is in turn founded on a theory of irrationals which we can credit to Theaetetus on the authority of Eudemus, we can hardly remain in doubt where the thoughts of that first audience would turn when they heard the narrative of Theodorus' lesson and the boys' definition of incommensurability. They must surely have felt themselves carried back to the beginning of an intellectual journey of great distinction. On Szabó's picture, it will be recalled, they would be savoring the naïveté of a boy who thinks that he and his companion have made a discovery when really they were deliberately led by Theodorus to work out for themselves a couple of simple classifications long familiar to mathematicians.[68] So read, the scene as Plato composed it would indeed be, to use Szabó's own comparison, a riddle to match the smile of La Gioconda. But then, astonishingly, Szabó simply declines to discuss the testimony of Eudemus concerning Theaetetus' historical involvement with the higher irrationals.[69]

The significance of Eudemus' testimony, to repeat, is that it enables us to see the definition in the dialogue as pointing forward to a large body of mature work in the theory of irrationals and in solid geometry: we can form some picture of the historical process which gave rise to Eudemus' more general statement (*apud* Proclus, *In Euclidis Elementa,* I 66, 14–18 Friedlein) that Theaetetus was one of those by whom "the theorems were increased in number and brought into a more scientific arrangement."[70]

The question on which Pappus and the scholiast disagreed can now take second place as largely a question of transmission: the question, that is, who put *Elements* X 9 into its present form in the process of transmission and adaptation which incorporated Theaetetus' original work into Books X and XIII of the *Elements.* It is acknowledged that the proof of X 9 as we have it must be a later addition.[71] Among those who could have lent a hand in the course of transmission we hear of one Hermotimus of Colophon who "pursued further the investigations already begun by Eudoxus and

[65]The evidential basis for the latter attribution, which in all probability goes back to Eudemus also, is Scholium 1 to Euclid, *Elements* XIII, Suidas *s.v.* Θεαίτητος; see Sachs, *Die fünf platonischen Körper,* Philologische Untersuchungen, Heft 24 (Berlin: Weidmann, 1917); Heath, *Greek Mathematics,* Vol. I, pp. 158–162; von Fritz, "Theaitetos," pp. 1363 ff.

[66]Van der Waerden, *Science Awakening,* pp. 173–174.

[67]Cf. also *Epinomis* 990d.

[68]Szabó, *Anfänge,* pp. 104–111.

[69]Szabó, "Theaitetos," p. 344; *Anfange,* pp. 103–104; similarly with Theaetetus' solid geometry, Szabó, "Theaitetos," p. 345.

[70]This and the following quotations from Proclus are given in the translation of Morrow, *Proclus.*

[71]See n. 64 above.

Theaetetus" (Proclus, *In Euclidis Elementa* I 67, 20–22), and later Euclid himself, in composing the *Elements,* is credited with "systematizing many of the theorems of Eudoxus, perfecting many of those of Theaetetus, and putting in irrefutable demonstrable form propositions that had been rather loosely established by his predecessors" (Proclus, *In Euclidis Elementa* 68, 6–10). But the central idea of X 9 must be Theaetetus' own, since it is fundamental to Book X as a whole.

Finally, a remark about the brief allusion at the end of Theaetetus' story (148b) to an extension of his definition to solids. Nothing corresponding to this is to be found in Book X, and it remains slightly obscure what the details would be.[72] But of one thing we may now be sure: it is not, as Vogt suggested, an unfounded analogy inserted by Plato to indicate Theaetetus' youthful haste.[73] The only reasonable explanation of so brief an allusion is that it is a further trace of Theaetetus' mathematics, confirming Plato's intention to commemorate a historical achievement.

DEFINITION AND DIALECTIC

There is an often-quoted remark of Reidemeister's which poses the question whether Theaetetus' reputation as a mathematician is not just a legend that crystallized around the character in Plato's dialogue.[74] I have been making a case for the opposite view, that a proper dramatic appreciation of the character in the dialogue is only to be gained by recognizing that the "legend" is genuine history, and recent history at that, framed by Plato between the backward reference in the prefatory encomium and the forward reference in the narrative of Theodorus' lesson. I have been endeavoring to assemble and assess the fragments of evidence which are all we have to fill in the substance of the intellectual achievements thus commemorated in the dialogue. The question that now remains is what relevance the mathematical episode has for the main business of the dialogue—the philosophical inquiry into knowledge.

This further question has not received the careful attention it deserves. Historians of mathematics have tended to isolate the episode from its wider context, as if the commemorative function was its sole dramatic purpose; van der Waerden expressly remarks that it "gives the impression of having been dragged in," that "it does not fit very well the philosophical discussion it has to introduce."[75] On the other hand, scholars who have looked for a philosophical moral have tended to find it in some purely literary, symbolic connection with other parts of the dialogue.

According to one suggestion of this kind, the connection is that in the dialogue "knowledge turns out, whatever unit of comparison we employ, to be incommensurable with opinion"[76]—a rather imprecise metaphor of which there is absolutely no sign in Plato's text. Other correspondences have been sought between Theaetetus' powers and the active and passive powers which figure in the Heraclitean theory of

[72] See the anonymous commentator 42, I ff.; Sachs, *De Theaeteto,* pp. 56–57; Heath, *Greek Mathematics,* Vol. I, p. 212; von Fritz, "Theaitetos," p. 1360; van der Waerden, "Die Arithmetik," pp. 237–238, 246, *Science Awakening,* pp. 166–167.

[73] Vogt, "Entdeckungsgeschichte," pp. 115–117, 127.

[74] Kurt Reidemeister, *Das exakte Denken der Griechen* (Hamburg: Claasen & Goverts, 1949), p. 24, basing himself on the fact that Pappus has no direct knowledge of Theaetetus' work. Reidemeister's remark is quoted by Szabó, "Der mathematische Begriff," p. 230, n. 27, with the comment, "Je mehr ich die voreuklidische Mathematik der Griechen kennenlerne, umso berechtigter scheint mir dieser Zweifel" (modified in Szabó, "Theaitetos," pp. 309, 345); it is quoted by van der Waerden, "Die Arithmetik," p. 248, with puzzled surprise that anyone can remain so skeptical.

[75] Van der Waerden, *Science Awakening,* pp. 142, 166.

[76] Robert S. Brumbaugh, *Plato's Mathematical Imagination* (Bloomington: Indiana University Press, 1954), p. 40; a variant version of the same idea is put forward by Gwynneth Matthews, *Plato's Epistemology* (London: Faber, 1972), p. 20.

perception at 156a ff.;[77] or again, between the former, lines commensurable in square only, and the primary elements of which Socrates dreams at 201e–202b, which in themselves cannot be expressed by a *logos* (account), whereas combinations of them (syllables) are expressible.[78] Now certainly, Plato is well able to enjoy a structural correspondence of this kind. But it would be uncharacteristic of him to let it become a substitute for serious philosophical content. Whatever the symbolic connections, we need a moral of greater consequence if we are really to integrate Theaetetus' story into the philosophical discussion.

Let us go back to the story and its setting within the methodological preliminaries to the discussion of knowledge. As already noted, the story illustrates a point about definition and examples: Socrates wants a definition of knowledge, not examples, and Theaetetus volunteers his mathematical definition to show that he has now grasped the true nature of the Socratic question What is knowledge? It is thus the definition which is highlighted as the climax of the story. Equally, it is the definition which calls forth Socrates' strongest commendation (148b) and which is set up as a model for Theaetetus to follow in answering the question about knowledge (148d). In all these ways the definition is the dramatic focus of the scene. What, then, we must ask, is the role of a definition in a Socratic discussion of the type which follows in the dialogue?

Every student of ancient philosophy knows that one of the things that may safely be attributed to the historical Socrates is the search for general definitions of ethical terms (see Aristotle, *Metaphysics* 1078b 17–30). What is less often remarked upon is the significance of the fact that the Socratic search for a definition begins with a definition.

A Socratic discussion of the type exemplified in Plato's early dialogues and, on a larger scale, in the first part of the *Theaetetus* (151e–183c) begins with Socrates' interlocutor proposing a definition which is then tested for validity. It has to be seen whether the definition is compatible, on the one hand with such examples of the problematic concept as may be volunteered or admitted by the interlocutor, on the other with any general beliefs or principles the interlocutor may have that bear upon the subject of discussion. Whenever, under the pressure of Socrates' questioning, an inconsistency comes to light in the interlocutor's overall position, some appropriate adjustment has to be made: the definition is modified or abandoned, or it is maintained at the price of jettisoning ideas incompatible with it, however plausible or commonsensical they may seem to be. As the interlocutor is brought gradually to see where his thesis leads—and it is important that the full implications of a definition are not apparent straight off—he has to reflect at each stage whether to go on with it and how far he can honestly revise other beliefs to arrive at a coherent overall theory. In other words, for the Socratic method of dialectic a definition is in the first instance a starting point for investigation, the worth of which will be proved only over the full range of inquiry to which it leads.[79]

[77] Kenneth M. Sayre, *Plato's Analytical Method* (Chicago/London: University of Chicago Press, 1969), pp. 60–61, 95; Brown, *"Theaetetus,"* pp. 376 ff. (although I am not persuaded by Brown's interpretation [see also notes 55, 58, 60 above], I salute his attempt, unique in the literature on the dialogue, to treat the mathematical and philosophical aspects of our passage with equal seriousness for the sake of a satisfactory integration of the two).

[78] Diès, *Theaetetus,* p. 128; M. F. Burnyeat, "The Material and Sources of Plato's Dream," *Phronesis,* 1970, *15*:105–106. This comparison has in its favor the fact that in expounding his dream of elements and syllables Socrates uses words like ἄλογος and ῥητός, which also occur as key terms in mathematical contexts dealing with irrationals.

[79] A detailed defense of this view of the Socratic method, with special reference to the first part of the

Now the most extended, the most highly structured specimen we possess of the Socratic method at work in this type of discussion is the elaboration and critique of the thesis that knowledge is perception in the first part of the *Theaetetus*—none other than the discussion which our mathematical passage introduces, devoted to the definition of knowledge as perception which is Theaetetus' eventual answer to Socrates' request to do for knowledge what he and his companion had done for incommensurability. This is ample justification for us to view the model mathematical definition in the same methodological perspective, as a starting point for investigation rather than a finished result or completion.

Nothing, after all, is said in the dialogue about the boys proving their proposition[80] or putting it to work in any mathematically interesting way. The proof,[81] the applications, were yet to come, as Plato's audience would know, and without a proof a mathematical proposition lacks much of its significance. Thus within the dialogue, confining ourselves to the details actually recounted there, Theodorus' contribution takes the palm, being furnished with proofs, and Theaetetus is still very much the junior partner. Which is as it should be, given the dramatic situation. The mathematical importance of his definition will emerge only in the future accomplishments which Socrates is said to have predicted (142d).

What we can detect in the dialogue is an approach to the handling of irrational quantities which was to be seminal for Theaetetus' mature theory—here I quote van der Waerden's comment on the treatment of the higher irrationals in *Elements* X:

> All these proofs are based on one fundamental idea which runs as a guiding thread through the entire book: *to prove properties of any type of line, one constructs a square on this line and one investigates the properties of this square.* For instance, to prove that a binomial can not be a medial, it is shown that the square on a binomial can not be a medial area.
>
> Properly speaking this basic idea already turns up in the first part of Book X and in the dialogue *Theaetetus,* for Theaetetus derived the incommensurability of certain line segments from the ratio of their squares.[82]

If this is on the right lines, it is in a seminal approach to the handling of irrational quantities, quite as much as in the particular theorem of *Elements* X 9, that we should look for the connection between the definition in the dialogue and the achievements for which Theaetetus came to be esteemed; it would be no objection that working out the theory may have involved him in some modification of the original thought. In any case, however the details of the connection are to be drawn in, what matters for our purposes is the evidence we have assembled of a substantial contribution by Theaetetus to the impressive theory of irrationals in Book X of Euclid's *Elements.* Given that, we can read the passage 147c–148b as celebrating less than a great step forward in the annals of mathematics, yet much more than a schoolboy's exercise. The scene before us is the birth of a highly fruitful idea, a youthful beginning, not a completion, to a far-reaching study of irrational quantities.[83]

Theaetetus, is given in my paper "Examples in Epistemology: Socrates, Theaetetus and G. E. Moore," *Philosophy,* 1977, *52*: 381–398.

[80] Hence there is no reason to follow Sayre, *Plato's Analytic Method,* pp. 57–59, in calling it an example of *knowledge,* i.e., of the subject proposed for the dialogue to discuss.

[81] Which, of course, we do not possess; for suggestions, see Sachs, *De Theaeteto,* pp. 52 ff.; von Fritz, "Theaitetos," pp. 1359–1360; van der Waerden, "Die Arithmetik," pp. 205–206, 236–237, *Science Awakening,* pp. 166–167.

[82] Van der Waerden, *Science Awakening,* p. 169.

[83] It is, of course, impossible to say whether the study continued to be in any sense a joint effort with the younger Socrates, although, as already noted, their early collaboration is emphasized again at *Sophist* 218b (cf. *Politicus* 266a 6–7).

So read, the passage leads naturally on to the main philosophical discussion.[84] The definition of knowledge as perception is also a beginning and pivot for a systematic investigation, an epistemological investigation the interest of which is not lessened by its negative outcome, the eventual refutation of the definition. The parallel between the philosophical definition and its mathematical model has to do with the value of a definition in giving impetus and direction to inquiry. It is not any and every definition which can do this; that a definition states necessary and sufficient conditions for the application of some concept is no guarantee of its leading further, not even of its being genuinely explanatory. Thus it is important that the rule found by Theaetetus and his companion not only gives a direct and straightforward method of generating both Theodorus' examples and an infinite series of further cases; it also offers a rubric under which the properties of incommensurable lines can be systematically investigated and general theorems proved.[85] For the purposes of the dialogue Plato's appeal to an actual example of fruitful definition in mathematics brings out in a peculiarly forceful way that, besides giving necessary and sufficient conditions, a definition should be of *systematic* interest.[86]

CONCLUSION

I have no doubt that in future histories of mathematics Theaetetus' story will continue to be told, notwithstanding Szabó's efforts to discredit it. It is a good story, and a credible story—provided one attends soberly to what is claimed and what is not claimed for Theodorus and Theaetetus respectively, and provided one is sensitive to its dramatic function within the dialogue. It is also an indispensable story, without which we would have no idea how to fill the gap between the Greeks' first encounter with incommensurability and the incorporation of incommensurable magnitudes into a general theory of proportions by Eudoxus, an event which took place very close in time to Plato's writing of the *Theaetetus*.[87]

What I have tried to do in this paper is, first, to vindicate the essential historicity of the story rather more carefully than has been done hitherto, making particular use of the results of ancient scholarship; and second, to bring out the philosophical sense of the scene, its contribution to the methodological preliminaries to the inquiry into knowledge. My essential claim has been that these are not two separate and independent tasks.[88]

[84] Mansfeld, "Notes," p. 114, connects the passage rather with its more immediate sequel, the account of Socrates' art of intellectual midwifery (148e–151d). This is because he accepts Szabó's view that Theaetetus' achievement is at best a rediscovery (see n. 5 above) and puts Theodorus' lesson parallel to Socrates' maieutic questioning. The parallel is at least doubtful, but the main objection is the sum of objections to Szabo's overall interpretation.

[85] Cf. the anonymous commentator, 44, 43 ff., who sees the merit of the definition in terms of gains in clarity and generality.

[86] One may compare here an earlier discussion of definition at *Meno* 75b ff.: two definitions of figure are contrasted, one identifying it as what invariably accompanies color, the other and more favored saying that figure is the limit of a solid. Socrates does not say much about why he prefers the latter definition, but one reason may be the way it is tailored to fit into a systematic, orderly investigation of geometrical entities; for a concern with the proper organization of inquiry runs all through the *Meno*, coming to the fore at 71ab, 81d, 86d ff.

[87] This consideration is rightly urged against Szabó, "Theaitetos," by Kurt von Fritz, *Platon, Theaetet und die antike Mathematik* (2nd ed.; Darmstadt: Wissenschaftliche Buchgesellschaft, 1969), Anhang, p. 73, n. 10 (but Szabó, *Anfänge*, pp. 131 ff., has his own views on what can be credited to Eudoxus).

[88] Note added in proof, Sept. 1978: This paper was completed and accepted for publication before I had access to Wilbur Richard Knorr, *The Evolution of the Euclidean Elements* (Dordrecht/Boston: Reidel, 1975). Knorr's massive reconstruction of the early history of incommensurability includes a careful

analysis of the *Theaetetus* passage which agrees both in general tenor and on many points of detail with that offered here. We agree also in disagreeing with Szabó, whose reading of the passage continues to be influential (for a recent example, see Hans-Joachim Waschkies, *Von Eudoxos zu Aristoteles,* Amsterdam: Grüner, 1977, p. 80 n.10). This happy convergence of independent opinions breaks down, however, on one important issue: the sentence at 147d 6, ἐν δὲ ταύτῃ πως ἐνεσχέτο.

It is not too much to say that large chunks of Knorr's impressive rewriting of the history of early Greek mathematics stand or fall by the thesis that this sentence means "but in this one [*sc.* the 17-foot power] for some reason he encountered difficulty" (Knorr, p. 62). That is, of the three versions distinguished above Knorr adopts reading (c), with the additional feature that ἐν ταύτῃ is given the specific meaning "in this δύναμις." Knorr insists on the specific meaning, as opposed to vaguer expressions like "at this point," so as to require that Theodorus came to a standstill at 17 because of a difficulty *at* 17, not (as has often been proposed) because of a difficulty soon afterwards at 19 (*ibid.*, pp. 81–83). And he offers a method of proof using Pythagorean number triples which both necessitates a case-by-case treatment and fails at 17 (*ibid.*, Ch. 6).

As stated the argument is less than conclusive, though it seems to have convinced at least one reviewer (Sabetai Unguru in *History of Science,* 1977, *15*:217). There is nothing illogical or objectionable about saying "At the 17-foot square he came to a standstill because of a difficulty just ahead at 19" ($\sqrt{18}$ is an uninteresting case, as it reduces to $3 \times \sqrt{2}$). After all, the previous sentence is naturally taken, as by the anonymous commentator (34, 15–28) and modern readers generally, to imply that Theodorus successfully effected the incommensurability proof for the case of 17 (οὕτω must resume ἔγραφε . . . ἀποφαίνων). In denying this, Knorr would have done better to appeal to philological evidence that in conjunction with ἐνέχομαι the preposition ἐν may introduce that by which someone is entangled or held up (see Liddell-Scott-Jones, *Greek-English Lexicon,* sv ἐνέχω). In other words, he could translate "*by* this one he was held up." But the dispute over where the difficulty is to be located is beside the point unless it is shown that ἐνέσχετο here does mean "got entangled" or "stopped because of a difficulty."

Against my evidence (above) that this is not the meaning of the verb here, Knorr has nothing to offer but Hackforth's now widely accepted but mistaken assertion that it is what the verb normally means. The truth is, as Mansfeld observed (n. 56 above), it is what the verb often (not always) signifies *in context,* not its intrinsic meaning. The idea of a difficulty or encumbrance has to come from the context; it is not brought to the context by the verb as part of its semantic contribution to the sentence in which it is used. Thus to vindicate his translation Knorr would have to find some implication in the context, *independent* of the occurrence of ἐνέχομαι, to the effect that the 17-foot square was a source of difficulty. And this he cannot do.

On the contrary, if the context implies anything on the matter, it rather implies that there was no difficulty at 17. The very next sentence includes the phrase "since the δυνάμεις [*sc.* the δυνάμεις with incommensurable sides—see above; Knorr, p. 68, agrees] were turning out to be unlimited in number." How could Theaetetus say such a thing if Theodorus' proofs had just broken down at the 12th example in the series from 3 on? Knorr has an answer (p. 85 and Ch. 6): Theodorus' proofs were such as to show, first, that a square with area given by any number of the form $4N + 3$ (e.g., 3,7,11,15,19 . . .) has its side incommensurable with unity; second, that the same holds for any number of the form $8N + 5$ (e.g., 5,13,21,29 . . .), and so on. *Each* of the examples mentioned in the dialogue stands for an infinite class of cases falling under the same proof. Failure comes at 17 because this is of the form $8N + 1$, which fits 9, 25, and all odd square numbers; only if the number is not a square number does it give rise to incommensurability—and this additional condition is, of course, the one that Theaetetus took as primary for his generalization. Ingenious—but we must ask whether Plato could expect his reader to understand, from the phrase "since the δυνάμεις were turning out to be unlimited in number," that Theodorus' proof method must have been such as to yield an infinity of cases *each* time it was successfully applied. I find it difficult to think that he could. But if not, Theaetetus' supposition of an infinite or indefinite number of cases of incommensurability looks incompatible with the idea that ἐνέσχετο refers to a difficulty or entanglement, whether at 17 or at 19. One ambiguous sentence in the dialogue is not a sound historical basis for Knorr's confident speculations about the character of Theodorus' proofs and ensuing developments in the study of incommensurability.

A further difficulty is that if Theodorus proceeded in the manner described, the move to Theaetetus' generalization would be obvious. Knorr agrees (p. 86). Theaetetus did not discover the theorem he states in the dialogue, but in later life he proved what for Theodorus had remained unproven conjecture. But this too is contrary to the indications of the dialogue, where the general definition of incommensurability is presented as something the boys themselves thought of seeking, and found (147d 7, e 2–3). It is no good professing to take the dialogue with the utmost seriousness as historical evidence and then ignoring vital bits of the evidence it supplies.

I admire Knorr's work. Its impact will be felt in every department of the study of Greek mathematics. But his treatment of evidence is not always as sober as it should be. (One last, small example: readers unfamiliar with such matters should not suppose that Knorr has anything but the most nugatory grounds for asserting [pp. 37 and 55, n. 44] that Theodorus did not begin his work in geometry until after 430 B.C.—when he would be 30 or 40 years old.) Plato's rendering of the young Theaetetus' story has the restraint of an Attic grave stele. If we respect that restraint, we will be content with what he has seen fit to tell us.